

# Categories of modules and their deformations

Romie Banerjee

## Abstract

We develop an obstruction theory for lifting compact objects to the stable  $\infty$  category of quasi-coherent modules over a derived geometric stack  $X$  from the category of modules over its underlying classical stack  $X^{cl}$ . The obstructions live in Andre-Quillen cohomology.

## 1 Introduction

The derived category of quasi-coherent modules over a scheme or an algebraic stack is usually very badly behaved in the sense that it is not controlled by a small data. In certain cases it is possible to find a set of compact generators of the derived category of modules in question. For example, if  $R$  is a commutative ring then the triangulated category  $D(R)$  of chain complexes of  $R$ -modules modulo weak equivalences of chain cohomology isomorphisms is compactly generated. Similar thing is true of the unbounded derived category of quasi coherent modules over a quasi-compact separated scheme [4]. In general not all algebraic stacks have this property. Stable homotopy theory gives rises to more sources of interesting triangulated categories. For any  $E_\infty$ -ring spectrum  $A$  the derived category of  $A$ -modules is compactly generated. For any derived scheme, formed by gluing derived affine schemes  $\mathrm{Spec}(A)$  along Zariski maps of  $E_\infty$ -rings, the derived category of quasi-coherent modules form a compactly generated triangulated category [1]. We are interested in the triangulated category of the derived category of quasi-coherent modules over any general derived algebraic stack. Throughout this paper we think of derived algebraic stacks, once rigidified, as being equivalent to cosimplicial connective  $E_\infty$ -rings.

Given a derived  $\infty$ -stack  $X$ , we want to study the stable  $\infty$ -category of quasi-coherent modules over  $X$ . If  $X$  is an algebraic stack, i.e.  $X$  admits an atlas by simplicial derived affine scheme  $U_\bullet$ , we get a cosimplicial stable  $\infty$ -category  $\mathrm{Mod}(U_\bullet)$ . The stable  $\infty$ -category modules over the stack  $X$  is the totalization  $\mathrm{Tot}(\mathrm{Mod}(U_\bullet))$ .

Let  $\mathfrak{C}_\infty$  be the category of connective  $E_\infty$  rings. Another way to approach this is to consider the stack  $QC$  considered as a moduli functor

$$QC : \mathfrak{C}_\infty \rightarrow \mathrm{Pr}_{st-\infty}$$

where the right side is the  $\infty$ -category of presentable stable  $\infty$ -categories, so that  $QC(A) = \mathrm{Mod}(A)$  and  $QC$  takes a map of modules  $f : A \rightarrow B$  to the functor  $- \otimes_A B$ . This naturally extends to a functor between  $\infty$ -categories.

The desired object, i.e. the  $\infty$ -category of quasi-coherent modules over any  $\infty$ -stack  $X : \mathfrak{C}_\infty \rightarrow SSet$  is  $\text{Hom}_{\infty\text{-stacks}}(X, QC)$ , the hom space in the  $\infty$ -topos of  $\infty$ -stacks.

If  $A$  is a connective  $E_\infty$  ring which admits a postnikov tower decomposition and  $\mathcal{M}$  an  $\infty$ -stack which admits a cotangent complex and is infinitesimally cohesive [6], lifting a family of objects classified by  $\mathcal{M}$  on the ordinary affine scheme  $\text{Spec } \pi_0 A$  to the derived affine scheme  $\text{Spec } A$  is a problem in deformation theory. It is controlled by the cotangent complex of the stack  $\mathcal{M}$ . Associated to any derived algebraic  $\infty$ -stack  $X$  there is an ordinary algebraic  $\infty$ -stack  $X^{cl}$  which admits an atlas of a cosimplicial ordinary ring obtained by taking sectionwise  $\pi_0$  of the the atlas of  $X$ . We can think of  $X$  as an infinitesimal extension of the underlying  $X^{cl}$ .

Let  $X$  be a derived  $\infty$ -stack. Let  $X^{cl}$  be it's associated classical (non-derived stack). There is a natural map  $i : X^{cl} \rightarrow X$ . The induced map on derived categories:

$$D_{qc}(X) \rightarrow D_{qc}(X^{cl})$$

Given arbitrary  $x$  in  $D_{qc}(X)$  and a perfect  $u$  in  $D_{qc}(X^{cl})$ , with a map  $u \rightarrow x$  we would like to find cohomological obstructions for lifting  $u$  to a perfect module  $\tilde{u}$  over  $X$  and a map  $\tilde{u} \rightarrow x$  over  $X$  which restricts to  $u \rightarrow x$  over  $X^{cl}$

The main result is

**Theorem 1.1.** *Let  $X$  be a perfect derived algebraic  $n$ -stack for some  $n$  and let  $\tilde{X}$  be a square-zero extension of  $X$ . Let  $x : \tilde{X} \rightarrow QC$  be a complex of quasi-coherent modules over  $\tilde{X}$  and let  $u : X \rightarrow QC^\omega$  be a complex of perfect modules over  $X$ , along with a map  $u \rightarrow x$  in  $QC(X)$ .*

- *Then there exists an obstruction theory for deforming  $u$  to a  $\tilde{u} : X \rightarrow QC^\omega$ . The space of deformations is isomorphic to  $\Omega \text{Hom}_{\mathcal{O}_X}(\alpha^* L_{QC^\omega}, N)$  with loops based at the trivial derivation.*
- *If this space is non-empty and  $\tilde{u}$  is a deformation of  $u$ , then there exists a perfect module  $y_\beta : X \rightarrow QC^\omega$  along with maps  $\beta : u \rightarrow y_\beta$  and  $y_\beta \rightarrow x$  in  $QC(X)$  such that the triangle commutes in  $QC(X)$*

$$\begin{array}{ccc} u & \xrightarrow{\quad} & x \\ & \searrow \beta & \nearrow \\ & y_\beta & \end{array}$$

*There is an obstruction theory for lifting  $\beta$  to  $\tilde{\beta} : \tilde{u} \rightarrow \tilde{y}_\beta$  such that  $\tilde{u} \rightarrow \tilde{y}_\beta \rightarrow x$  is a deformation of  $\alpha : u \rightarrow x$ .*

*More precisely, there exists a moduli functor  $\mathcal{G} : \Omega_{u, y_\beta} QC_{/X \times X}$  and an cocycle in the Andre-Quillen cohomology*

$$\alpha(u, y_\beta) \in \text{Hom}_{\mathcal{O}_X}(\beta^* L_{\mathcal{G}}, N)$$

such that, if  $\alpha(u, y_\beta) = 0$  there exists a lift  $\tilde{\beta}$ . The space of all such deformations is isomorphic to

$$\Omega \mathrm{Hom}_{\mathcal{O}_X}(\beta^* L_{\mathcal{G}}, N)$$

where the loops are based at the trivial derivation.

## Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Derived <math>\infty</math>-stacks: Overview</b>	<b>3</b>
2.1	The quasi-coherent $\infty$ -stack . . . . .	4
<b>3</b>	<b>Deformation Theory</b>	<b>5</b>
3.1	Infinitesimal Extensions of $\infty$ -stacks . . . . .	7
3.2	Cotangent complexes of $\infty$ -stacks . . . . .	8
<b>4</b>	<b>Obstruction Theory</b>	<b>10</b>
<b>5</b>	<b>Moduli of compact objects of <math>QC(X)</math></b>	<b>11</b>
<b>6</b>	<b>Proof of the Main Theorem</b>	<b>14</b>
	<b>References</b>	<b>17</b>

## 2 Derived $\infty$ -stacks: Overview

In this section we give a brief introduction to geometric  $\infty$ -stacks. The reader may find the necessary details on  $\infty$ -categories and  $\infty$ -topoi in [3].

Let  $\mathcal{C}^{op}$  denote a presentable  $\infty$ -category (connective  $E_\infty$  rings, or simplicial commutative rings) with an topology  $\tau$  on  $\mathcal{C}$ . A derived  $\infty$ -prestack is a functor

$$\mathcal{F} : \mathcal{C}^{op} \rightarrow \mathcal{S}$$

$$\mathrm{Spec} A = \mathrm{Hom}_{\mathcal{C}^{op}}(A, -)$$

$\mathcal{F}$  is an  $\infty$ -stack if it satisfies Čech descent with respect to  $\tau$ ;  $X \in \mathrm{Fun}^L(\mathcal{C}^{op}, \mathcal{S})$  and  $\mathcal{F}$  takes the Čech nerve of any  $\tau$ -cover  $U \rightarrow X$  to a limit diagram.

$\mathcal{F}$  is an algebraic  $\infty$ -stack if there is a cosimplicial object  $A_\bullet \in (\mathcal{C}^{op})^{N(\Delta)}$  and  $\mathcal{F}(B) = |\mathrm{Hom}_{\mathcal{C}^{op}}(A_\bullet, B)|$ ,

$$\mathcal{F} = \mathrm{colim}_{\Delta^{op}} \mathrm{Spec} A_\bullet$$

in the  $\infty$ -category of  $\infty$ -stacks.

## 2.1 The quasi-coherent $\infty$ -stack

A quasi-coherent sheaf on a scheme  $X$  is a morphism of stacks  $X \rightarrow Mod$  from  $X$ , considered as a stack, into the canonical stack

$$Mod : Spec A \mapsto Mod_A$$

of modules which corresponds to the bifibration

$$T_{CRings} \simeq Mod \rightarrow CRings$$

from the tangent category of the the category of commutative rings to commutative rings.

This definition of quasi-coherent sheaves generalizes to any  $(\infty, 1)$ -topos, and over arbitrary  $\infty$ - sites. Let  $\mathcal{C}$  be symmetric monoidal  $\infty$ -category equipped with Grothendieck  $\infty$  topology such that  $\mathcal{C}^{op}$  is presentable. The tangent  $\infty$  category  $T(\mathcal{C}^{op}) \rightarrow \mathcal{C}^{op}$  is the bifibration whose fibers over an object  $A \in \mathcal{C}$  plays the role of the  $\infty$ -groupoid of modules over  $A$ , see [2].

Under the  $\infty$ -Grothendieck construction this corresponds to a  $(\infty, 1)$  presheaf

$$Mod_\infty : \mathcal{C}^{op} \rightarrow \widehat{Cat}_\infty$$

$$Mod_\infty : Spec R \mapsto Stab(\mathcal{C}_{/R}^{op})$$

where  $Spec R$  for  $R \in \mathcal{C}^{op}$  is the affine object in the geometry defined over  $\mathcal{C}^{op}$ , or directly in terms of test spaces

$$Mod_\infty : \mathcal{U} \mapsto Stab(\mathcal{C}^{\mathcal{U}/})$$

This makes  $Mod_\infty^{\mathcal{C}} \in Shv_{(\infty, 1)}^{\widehat{Cat}_\infty}(\mathcal{C}) = [\mathcal{C}^{op}, \widehat{Cat}_\infty]$ .

Let  $H = Shv_\infty(\mathcal{C})$  be the  $\infty$ -topos of  $\infty$ -stacks on  $\mathcal{C}$  and  $X \in H$  be an  $\infty$ -stack. The stable  $\infty$  category of quasi coherent modules over  $X$  is the Hom space in the  $\infty$ -topos  $H$ ;

**Definition 2.1.**

$$QC(X) = Hom_H(X, Mod_\infty) \tag{1}$$

Notice that  $H \subset [\mathcal{C}^{op}, \widehat{Cat}_\infty]$  as any  $\infty$ -groupoid is in  $\widehat{Cat}_\infty$ .  $QC(X)$  is computed using the Yoneda-Kan extension.

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{j} & P(\mathcal{C}) \\ \downarrow Mod & \swarrow & \uparrow Kan_j(Mod) \\ \widehat{Cat} & & \end{array}$$

$$Fun(\mathcal{C}^{op}, \mathcal{D}) \xrightleftharpoons[j]{Kan} Fun^R(P(\mathcal{C})^{op}, \mathcal{D})$$

By definition  $Kan(F)(Y) = \lim_{j(U) \rightarrow Y} F(U)$ , where  $Y = \text{colim}_{j(U) \rightarrow Y} j(U)$  in  $P(\mathcal{C})$ . The above adjunction is an equivalence of  $\infty$  categories; it follows

from the the standard adjunction  $\text{Fun}(\mathcal{A}, \mathcal{B}) \xrightleftharpoons[j]{\text{Kan}} \text{Fun}^L(P(\mathcal{A}), \mathcal{B})$  being an equivalence of  $\infty$  categories.

For a prestack  $X \in P(\mathcal{C}) = \text{Fun}(\mathcal{C}^{op}, \text{SSets})$ , suppose  $X = \text{colim}_\alpha j(\text{Spec} R_\alpha)$  the

$$\widetilde{QC}(X) = \text{Kan}_j(\text{Mod})(X) = \lim_\alpha \text{Mod}(\text{Spec} R_\alpha) = \lim_\alpha \text{Stab}(\mathcal{C}_{/R_\alpha}^{op}).$$

If  $X$  is an  $\infty$ -stack,  $QC(X)$  can be expressed as a limit similarly,

$$QC : \infty\text{-stacks} \xrightarrow{i^{op}} P(\mathcal{C})^{op} \xrightarrow{\widetilde{QC}} \widehat{\text{Cat}}$$

however since  $i^{op}$  doesn't preserve limits, it is not straightforward to show.

If  $X$  is a geometric  $\infty$ -stack (i.e. atlas by a simplicial object in  $\mathcal{C}$ ), we want to compute  $QC(X)$ .  $QC$  is the composition

$$QC : \text{geometric-}\infty\text{-stacks}^{op} \cong \mathcal{C}^\Delta \xrightarrow{i^{op}} P(\mathcal{C}^{op}) \xrightarrow{\widetilde{QC}} \widehat{\text{Cat}}_\infty$$

If  $A_\bullet \in \mathcal{C}^\Delta$  is the cosimplicial object such that simplicial object in  $\mathcal{C}$ ,  $\text{Spec}(A_\bullet)$  (or simply, the simplicial affine scheme) is an atlas for  $X$ , then

$$i(A_\bullet) = \text{Hom}_{\mathcal{C}^{op}}(A_\bullet, -)$$

That is, as an object in the prestack category  $i(A_\bullet)$  evaluates on objects in  $\mathcal{C}^{op}$  as the geometric realization

$$i(A_\bullet)(R) = |\text{Hom}_{\mathcal{C}^{op}}(A_\bullet, R)|.$$

or,  $i(A_\bullet) = \text{colim}_{\Delta^{op}} \text{Spec}(A_\bullet)$  in the  $\infty$  category of affine  $\mathcal{C}$ -schemes.

Therefore,

$$QC(A_\bullet) = \widetilde{QC}(i^{op}(A_\bullet)) = \widetilde{QC}(\lim_{\Delta} \text{Spec}(A_\bullet)) = \lim_{\Delta} \widetilde{QC}(\text{Spec} A_\bullet) = \text{TotMod}(A_\bullet) \quad (2)$$

where the limit/Tot is taken in the category of the stable presentable  $\infty$  categories.

### 3 Deformation Theory

In this section we describe the basic setup for doing deformation theory of geometric  $\infty$ -stacks. We will closely follow Lurie's DAG IV [2].

Let  $\mathcal{D}$  be a presentable  $\infty$  category, then the tangent category  $T_{\mathcal{D}}$  is the fiberwise stablization of the projection map

$$\text{Fun}(\Delta^1, \mathcal{D}) \rightarrow \text{Fun}(\{1\}, \mathcal{D}) \simeq \mathcal{D}$$

Roughly speaking, an object of the tangent bundle  $T_{\mathcal{D}}$  consists of a pair  $(A, M)$ , where  $A \in \mathcal{D}$  and  $M \in \text{Stab}(\mathcal{D}_{/A})$ ; here  $\text{Stab}$  is the stabilization construction applied to an  $\infty$  category. If  $\mathcal{D}$  is the ordinary category of commutative

rings(replace stabilization with abelianization) then the associated tangent category is equivalent to the category of modules; the objects are pairs  $(A, M)$ , where  $A$  is a commutative ring and  $M$  is a  $A$ -module. If  $\mathcal{D}$  is the  $\infty$ -category of  $E_\infty$ -rings or simplicial commutative rings then the tangent category recovers the categories of modules over such objects. Using this analogy, we can define a *module* over an object  $A$  to be an object of the fiber of the tangent category  $T_{\mathcal{D}}$  over  $\mathcal{D}$ , ie. the stable  $\infty$ -category  $T_{\mathcal{D}} \times_{\mathcal{D}} A \simeq \text{Stab}(\mathcal{D}_{/A})$ .

The *cotangent complex functor*  $L : \mathcal{D} \rightarrow T_{\mathcal{D}}$  is the left adjoint to the forgetful functor

$$T_{\mathcal{D}} \rightarrow \text{Fun}(\Delta^1, \mathcal{D}) \rightarrow \text{Fun}(\{0\}, \mathcal{D}) \simeq \mathcal{D}$$

such that the cotangent complex  $L_A$  of an object  $A$  is in  $\text{Stab}(\mathcal{C}_{/A})$ . In other words, the composition

$$\mathcal{C} \xrightarrow{L} T_{\mathcal{D}} \rightarrow \text{Fun}(\Delta^1, \mathcal{D}) \rightarrow \text{Fun}(1, \mathcal{D}) \simeq \mathcal{D}$$

is the identity functor.

The absolute cotangent complex functor  $L : \mathcal{D} \rightarrow T_{\mathcal{D}}$  is defined to be the composition

$$\mathcal{D} \rightarrow \text{Fun}(\Delta^1, \mathcal{D}) \rightarrow T_{\mathcal{D}}$$

where the first map is the given by the diagonal embedding and the second map is the left adjoint to the forgetful functor  $G : T_{\mathcal{D}} \rightarrow \text{Fun}(\Delta^1, \mathcal{D})$

$$\begin{array}{ccc} T_{\mathcal{D}} & \xrightarrow{G} & \text{Fun}(\Delta^1, \mathcal{D}) \\ & \searrow p & \swarrow \text{ev}_1 \\ & \mathcal{D} & \end{array}$$

Since the diagonal embedding is left adjoint to the evaluation map  $\text{Fun}(\Delta^1, \mathcal{D}) \rightarrow \text{hbox}(\{0\}, \mathcal{D})$  the absolute cotangent complex functor is left adjoint to the composition  $T_{\mathcal{D}} \rightarrow \text{Fun}(\Delta^1, \mathcal{D}) \rightarrow \text{Fun}(\{0\}, \mathcal{D})$ .

The fiber of the tangent bundle  $T_{\mathcal{D}}$  over  $A \in \mathcal{D}$  can be identified with the stable envelope  $\text{Stab}(\mathcal{D}_{/A})$ . Under this identification the cotangent complex  $L_A \in \text{Stab}(\mathcal{D}_{/A})$  corresponds to the image of  $\text{id}_A \in \mathcal{D}_{/A}$  under the suspension functor

$$\Sigma^\infty : \mathcal{D}_{/A} \rightarrow \text{Stab}(\mathcal{D}_{/A}).$$

The *trivial square zero extension* of  $A \in \mathcal{D}$  along a  $A$ -module  $M$ , denoted by  $A \oplus M$  is the image of the  $M$  under the functor

$$\Omega^\infty : \text{Stab}(\mathcal{D}_{/A}) \rightarrow \mathcal{D}_{/A} \rightarrow \mathcal{D}$$

Given an object  $A \in \mathcal{D}$  and a  $A$ -module  $M \in T_{\mathcal{D}} \times_{\mathcal{D}} \{A\}$ , a *derivation* of  $A$  into  $M$  is a map  $\eta : L_A \rightarrow M$  in the  $\infty$ -category  $T_{\mathcal{D}} \times_{\mathcal{D}} \{A\}$ . The derivation  $\eta$  equivalently gives a map from  $A$  to the trivial square-zero extension of  $A$  defined by  $M$  in the category  $\mathcal{D}$ ,

$$d_\eta : A \rightarrow A \oplus M$$

The derivation classified by the zero map  $L_A \rightarrow M$  (this is a stable category) corresponds a canonical section  $d_0 : A \rightarrow A \oplus M$  in  $\mathcal{D}$ . The *square-zero extension*

of  $A$  defined by the derivation  $\eta : L_A \rightarrow M$  is the pullback in the  $\infty$  category  $\mathcal{D}$ .

$$\begin{array}{ccc} A^\eta & \longrightarrow & A \\ \downarrow & & \downarrow d_0 \\ A & \xrightarrow{d_\eta} & A \oplus M \end{array}$$

Let  $f : \tilde{A} \rightarrow A$  be a morphism in  $\mathcal{D}$ . Then  $f$  is a square-zero extension if there exists a derivation  $\eta : L_A \rightarrow M$  and an equivalence  $\tilde{A} \simeq A^\eta$  in the  $\infty$ -category  $\mathcal{D}_{/A}$ .

The square-zero extension  $\tilde{A}$  will also be alternatively denoted by  $A \oplus_\eta \Omega M$ , so that  $A \oplus_0 \Omega M \simeq A \oplus M$ .

### 3.1 Infinitesimal Extensions of $\infty$ -stacks

Suppose  $A_\bullet$  is a cosimplicial object in  $\mathcal{C}^{op}$  and  $X = \operatorname{colim}_{\Delta^{op}} \operatorname{Spec} A_\bullet$  the associated algebraic stack. Then

$$\operatorname{Mod}_{\mathcal{O}_X} = \operatorname{Hom}_H(X, \operatorname{Mod})$$

We have seen how to compute this

$$\operatorname{Mod}_{A_\bullet} = \operatorname{Tot}_{[n] \in \Delta} \operatorname{Stab}(\mathcal{C}_{/A_n}^{op}) \simeq \operatorname{Tot}_{[n] \in \Delta} \operatorname{Mod}_{A_n}$$

Therefore a module over  $\mathcal{O}_X$  is an object in the totalization of a cosimplicial stable  $\infty$ -category. The 0-simplices of the Tot stable  $\infty$ -category are exactly  $A_0$ -modules + descent data, i.e.  $\mathcal{O}_X$ -modules. A  $\mathcal{O}_X$ -module  $N$  is a cosimplicial diagram of modules,  $N_n \in \operatorname{Stab}(\mathcal{C}_{/A_n}^{op})$ , and descent data. The trivial square zero extension defined by each  $A_n$ -module  $N_n$  is the image of  $N_n$  under the map  $\operatorname{ev}_0 \circ \Omega^\infty : \operatorname{Stab}(\mathcal{C}_{/A_n}^{op}) \rightarrow \mathcal{C}^{op}$ .

Let  $\operatorname{Stab}(\mathcal{C}_{/A_\bullet}^{op})$  denote the cosimplicial stable  $\infty$ -category induced by the cosimplicial diagram  $A_\bullet$ ; given a map  $A \rightarrow B$  in  $\mathcal{C}^{op}$  there is a naturally induced map of stable  $\infty$ -categories

$$\operatorname{Stab}(\mathcal{C}_{/A}^{op}) \rightarrow \operatorname{Stab}(\mathcal{C}_{/B}^{op}).$$

The limit of this diagram in the  $\infty$ -category of stable  $\infty$ -categories is the category whose objects consists of an object in each category and descent data required to glue them. In the general situation we will use the notation  $\mathcal{D}_{/A \rightarrow B}$  for the  $\infty$ -category  $\lim(\mathcal{C}_{/A} \rightarrow \mathcal{C}_{/B})$ . Therefore in our case of interest, the  $\infty$ -category  $\operatorname{Stab}(\mathcal{C}_{/A_\bullet}^{op})$  (where  $\operatorname{Stab}$  is taken level-wise) is the limit category  $\operatorname{Tot}_{[n] \in \Delta} \operatorname{Stab}(\mathcal{C}_{/A_{[n]}}^{op})$ .

We can apply the functor  $\Omega^\infty$  to the cosimplicial stable presentable  $\infty$ -category and compose with evaluation at  $\{0\} \in \Delta^1$ .

$$\operatorname{Stab}(\mathcal{C}_{/A_\bullet}^{op}) \xrightarrow{\Omega^\infty} \mathcal{C}_{A_\bullet}^{op} \xrightarrow{\operatorname{ev}_0} \mathcal{C}^{\Delta^{op}}$$

Let  $A_\bullet \in (\mathcal{C}^{op})^\Delta$  and  $N$  a  $A$ -module, that is an object in the totalization of the cosimplicial category  $\text{Stab}(\mathcal{C}_{/A}^{op})$ . The *trivial square-zero extension* of  $A_\bullet$  defined by  $N$  is the image of  $N$  under the map  $\Omega^\infty \circ \text{ev}_0$ . Denote this cosimplicial object in  $\mathcal{C}^{op}$  by  $A_\bullet \oplus N$ .

If  $X$  is the geometric  $\infty$ -stack whose atlas is the simplicial affine  $\mathcal{C}$ -scheme  $\text{Spec} A_\bullet$ , we'll denote the trivial square zero extension by  $\mathcal{O}_X \oplus N$ .

The absolute cotangent complex of a cosimplicial ring  $A_\bullet$  is the absolute cotangent complex of the associated geometric stack  $X = \text{colim}_{\Delta^{op}} \text{Spec} A_\bullet$ ,  $L_X$  (defined in the next section).

$L_X \in \text{Stab}(\mathcal{C}_{/A_\bullet}^{op})$ . For any  $\mathcal{O}_X$ -module  $N$ , a *derivation* of  $X$  into  $N$  is a map on the stable  $\infty$ -category  $\text{Stab}(\mathcal{C}_{/A_\bullet}^{op})$

$$\eta : L_X \rightarrow N$$

By adjunction, this is equivalent to giving a map  $A_\bullet \rightarrow A_\bullet \oplus N$  in  $\mathcal{C}^{\Delta^{op}}$ .

The *square-zero extension of  $A_\bullet$  defined by  $\eta$*  is the pullback in the  $\infty$ -category  $\mathcal{C}^{\Delta^{op}}$

$$\begin{array}{ccc} A_\bullet^\eta & \longrightarrow & A_\bullet \\ \downarrow & & \downarrow d_0 \\ A_\bullet & \xrightarrow{d_\eta} & A_\bullet \oplus N \end{array}$$

Denote the geometric  $\infty$ -stack defined by the atlas  $\text{Spec} A_\bullet^\eta$  by  $X \oplus_\eta [\Omega N]$ .

### 3.2 Cotangent complexes of $\infty$ -stacks

The *cotangent complex of an  $\infty$ -stack*. Let  $F$  be an  $\infty\mathcal{C}$ -stack, i.e. an object in  $\text{Fun}(\mathcal{C}^{op}, \text{SSet})$ . For  $A \in \mathcal{C}^{op}$  and  $M \in \text{Stab}(\mathcal{C}_{/A}^{op})$ . Let  $A \oplus M$  be the trivial square-zero extension of  $A$  by  $M$ . Let

$$x : \text{Spec} A \rightarrow F$$

be a  $A$ -point. Fix the following notation

$$X := \text{Spec} A$$

$$X[M] := \text{Spec}(A \oplus M)$$

The natural augmentation  $A \rightarrow A \oplus M$  gives a natural map of stacks  $X \rightarrow X[M]$ .

The *space of derivations* from  $F$  to  $M$  at  $x$  is defined by

$$\text{Def}_F(x, M) := \text{Hom}_{X/A_{\text{ffc}}}(X[M], F)$$

As  $M \mapsto X[M]$  is functorial in  $M$  is functorial in  $M$ , there is a well defined functor

$$\text{Def}_F(x, -) : \text{Mod}_A \rightarrow \text{SSet}$$

defined to be the homotopy fiber in the  $\infty$ -category of simplicial sets

$$\begin{array}{ccc} \text{Def}_F(x, M) & \longrightarrow & F(X[M]) \\ \downarrow & & \downarrow \\ \star & \xrightarrow{x} & F(X) \end{array}$$



The map  $X \rightarrow X[M]$  has a canonical section ( the zero derivation  $d_0 : A \rightarrow A \oplus M$ ). Therefore,  $\text{Def}_F(x, M)$  is a pointed space.

$F$  has a cotangent complex at  $x$  if the functor  $\text{Def}_F(x, M)$  is corepresented by a  $A$ -module  $L_{F,x}$ . The module  $L_{F,x} \in \text{Stab}(\mathcal{C}_{/A}^{op})$  is the cotangent complex of  $F$  at  $x$ .

The  $\infty$  stack  $F$  has an *absolute cotangent complex* if for any  $A \in \mathcal{C}^{op}$  and any  $x \in F(A)$ ,  $F$  has a cotangent complex  $L_{F,x}$  at  $x$  and for any commutative diagram in  $\infty\text{-stacks}/_F$

$$\begin{array}{ccc} \text{Spec} A & \xrightarrow{u} & \text{Spec} B \\ & \searrow x & \swarrow x' \\ & F & \end{array}$$

the natural morphism  $u^* L_{F,x'} \rightarrow L_{F,x}$  is an equivalence in  $\text{Stab}(\mathcal{C}_{/A}^{op})$ . In such a case denote the absolute cotangent complex of  $F$  by  $L_F$ . This a  $\mathcal{O}_F$ -module.  $L_F$  is an object in the stable  $\infty$ -category  $\lim_{\text{Spec} A \rightarrow F} \text{Stab}(\mathcal{C}_{/A}^{op})$ .

Suppose there is a map of  $\infty$ -prestacks  $F \rightarrow F'$ . Since  $A \oplus N \rightarrow A$  has a canonical section, given by the zero derivative,  $\text{Def}_F(x, N)$  is a pointed set. Denote by

$$\text{Def}_{F/F'}(x, -) : \text{Mod}_A \rightarrow \text{SSETS}$$

the homotopy fiber of the map

$$df : \text{Def}_F(x, -) \rightarrow \text{Def}_{F'}(x, -)$$

There is an alternate description of  $\text{Def}_{F/F'}(x, -) : \text{Mod}_A \rightarrow \text{SSETS}$ . Consider the functor  $G : \mathcal{C}\text{-stacks}/_{F'} \rightarrow \text{SSETS}$  which is the restriction of  $F$  to along the natural map  $\mathcal{C}\text{-stacks}/_{F'} \rightarrow \mathcal{C}\text{-stacks}$ . Then for a point  $x : \text{Spec} A \rightarrow F$ , there is a point  $x : \text{Spec} A \rightarrow G$  where  $\text{Spec} A$  is considered an object in the over-category  $\mathcal{C}\text{-stacks}$  via the map  $\text{Spec} A \rightarrow F \rightarrow F'$ . The relative deformation functor at  $x$ ,  $\text{Def}_{F/F'}(x, -)$  is then equivalent to the absolute deformation functor  $\text{Def}_G(x, -)$ .

$F \rightarrow F'$  has a relative cotangent complex at  $x$  if  $\text{Def}_{F/F'}(x, -)$  is corepresentable by an  $n$ -connective  $A$ -module  $L_{F/F',x}$  for some integer  $n$ .

$F \rightarrow F'$  has a *relative cotangent complex* if  $F \rightarrow F'$  has a relative cotangent complex at  $x$  for all points  $x$  and given a commutative diagram in  $\infty\text{-stacks}$

$$\begin{array}{ccc} \text{Spec} A & \xrightarrow{u} & \text{Spec} B \\ & \searrow x & \swarrow x' \\ & F & \end{array}$$

the natural morphism  $u^* L_{F/F',x'} \rightarrow L_{F/F',x}$  is an equivalence in  $\text{Mod}_A$ .

Suppose there is a sequence of maps of  $\infty$ -prestacks

$$F \rightarrow F' \rightarrow F''$$

and suppose the relative cotangent complex  $F'/F''$  exists, then there is an exact triangle in the stable  $\infty$ -category of  $F$ -modules

$$L_{F'/F''}|_F \rightarrow L_{F/F''} \rightarrow L_{F/F'}$$

in the sense that if either of the second or the third term exist then so does the other and the triangle.

## 4 Obstruction Theory

In this section we extend the Toën-Vessozzi [6] *obstruction theory* formalism for derived affine schemes to algebraic  $\infty$ -stacks.

Suppose  $d_\eta : X[M] \rightarrow X$  is a derivation, induced by a map  $\eta : L_A \rightarrow M$  in  $\text{Stab}(\mathcal{C}_{/A}^{op})$ . Define  $X_\eta[\Omega M] := \text{Spec}(A \oplus_\eta \Omega M)$ . Then the pullback square

$$\begin{array}{ccc} A \oplus_\eta \Omega M & \longrightarrow & A \\ \downarrow & & \downarrow d_0 \\ A & \xrightarrow{d_\eta} & A \oplus M \end{array}$$

means  $X_\eta[\Omega M]$  is the homotopy pushout  $X \coprod_{X[\Omega M]}^h X$  in the  $\infty$ -category of affine  $\mathcal{C}$ -schemes.

**Definition 4.1.** ([6]) *An  $\infty$ -prestack  $F$  has an obstruction theory if*

- (i)  *$F$  is infinitesimally cohesive*
- (ii)  *$F$  has a cotangent complex*

Geometric  $\infty$ -stacks always have an obstruction theory.

Suppose  $F$  has an obstruction theory then there exists a natural obstruction  $\alpha(x) \in \text{Hom}_{\text{Mod}_A}(L_{F,x}, M)$  for a  $A$ -point  $x : X \rightarrow F$  and  $X_\eta[\Omega M]$  as defined above. This cohomological (Andre-Quillen) obstruction vanishes iff the dotted arrow exists in the diagram

$$\begin{array}{ccc} & X_\eta[\Omega M] & \\ & \uparrow & \searrow x' \\ X & \xrightarrow{x} & F \end{array}$$

If  $\alpha(x) = 0$ , the space of lifts of  $x$ ,  $\text{Hom}_{X/Aff_{\mathcal{C}}}(X_\eta[\Omega M], F)$ , is isomorphic to  $\text{Hom}_{\text{Mod}_A}(L_{F,x}, \Omega M) \simeq$

$$\Omega \text{Hom}_{\text{Mod}_A}(L_{F,x}, M).$$

Is there a similar obstruction theory for lifting a family of object over an algebraic  $\infty$ -stack classified by a moduli stack  $F$  which has an obstruction theory? Suppose  $X = \text{colimSpec} A_\bullet$  (colimit in the  $\infty$ -category  $\mathcal{C}$ , i.e. the category of affine  $\mathcal{C}$ -schemes) where  $A_\bullet$  is cosimplicial  $\mathcal{C}^{op}$ -object. Let  $N \in \text{Stab}(\mathcal{C}_{/A_\bullet}^{op})$  be a  $A_\bullet$ -module and let  $A_\bullet^\eta$  be the square-zero extension of  $A_\bullet$  along a derivation  $\eta : L_X \rightarrow N$ . We want to find an obstruction for existence of the dotted arrow in

$$\begin{array}{ccc}
& \text{Spec}(A_\bullet^\eta) & \\
& \uparrow & \searrow x' \\
\text{Spec}(A_\bullet) & \xrightarrow{x} & F
\end{array}$$

where  $x$  is a  $X$ -point of  $F$ . It is clear from definitions that  $\text{Spec}(A_\bullet^\eta) \simeq \text{Spec} A_\bullet \coprod_{\text{Spec} A_\bullet \oplus N}^h \text{Spec} A_\bullet$ .

We need to verify that the following is an equivalence of simplicial sets when  $F$  is infinitesimally cohesive

$$F(A_\bullet^\eta) \simeq F(A_\bullet) \times_{F(A_\bullet \oplus N)}^h F(A_\bullet).$$

Here for any cosimplicial  $\mathcal{C}$ -object  $B_\bullet$ ,  $F(B_\bullet)$  is defined to be  $F(\text{colim}_{\Delta^{op}} \text{Spec} B_\bullet)$  using the Kan extension along the Yoneda map  $\mathcal{C} \rightarrow P(\mathcal{C})$ .

The following sequence of equivalences gives our desired equivalence.

$$\begin{aligned}
F(A_\bullet^\eta) &\simeq \text{Tot}_{[n] \in \Delta} F(A_{[n]}^\eta) \\
&\simeq \text{Tot}_{[n] \in \Delta} (F(A_{[n]}) \times_{F(A_{[n]} \oplus N_{[n]})}^h F(A_{[n]})) \\
&\simeq \text{Tot} F(A_\bullet) \times_{\text{Tot} F(A_\bullet \oplus N)}^h \text{Tot} F(A_\bullet) \\
&\simeq F(\text{colim} \text{Spec} A_\bullet) \times_{F(\text{colim} \text{Spec}(A_\bullet \oplus N))}^h F(\text{colim} \text{Spec} A_\bullet)
\end{aligned}$$

## 5 Moduli of compact objects of $QC(X)$

**Definition 5.1.** ([3]) *A object  $x$  in an  $\infty$ -category  $\mathcal{D}$  is compact if the functor  $\text{Hom}_{\mathcal{D}}(x, -) : \mathcal{D} \rightarrow \infty\text{-groupoids}$  commutes with small colimits;*

$$\text{Hom}_{\mathcal{D}}(x, \text{colim}_\alpha y_\alpha) \simeq \text{colim}_\alpha \text{Hom}_{\mathcal{D}}(x, y_\alpha).$$

$\mathcal{D}$  is compactly generated if there exists a family of compact objects  $\{x_\alpha\}_\alpha$  such that, any map  $X \rightarrow Y$  in  $\mathcal{D}$  is an equivalence if and only if  $\text{Hom}_{\mathcal{D}}(x_\alpha, X) \rightarrow \text{Hom}_{\mathcal{D}}(x_\alpha, Y)$  is a weak equivalence of simplicial sets for all  $\alpha$ .

A stable  $\infty$ -category  $\mathcal{D}$  is compactly generated if there is a family of compact objects such that  $y \in \mathcal{D}$  is the zero object iff  $\text{Hom}_{\mathcal{D}}(x_\alpha, y)$  is a contractible simplicial set for all  $\alpha$ . In other words, for any arbitrary  $y$  which is not the zero object, there is a non-zero map  $c \rightarrow y$  from some compact object  $c$ .

The  $\infty$ -stack of perfect quasi-coherent modules  $QC^{perf}$ . Consider the  $\infty$  functor considered as an object in  $P(\mathcal{C})$

$$\text{Mod} : \mathcal{C}^{op} \rightarrow \widehat{\mathcal{Cat}}_{\infty, st}$$

$$A \mapsto \text{Stab}(\mathcal{C}_{/A}^{op})^\omega.$$

For a map  $A \rightarrow B$  in  $\mathcal{C}$ , there is a map of  $\infty$  categories  $\text{Stab}(\mathcal{C}_{/A}^{op})^\omega \rightarrow \text{Stab}(\mathcal{C}_{/B})^\omega$  since compact object objects map to compact objects. This extends

to an  $\infty$ -functor.  $QC^{perf}$  is the  $\infty$ -stack (fppf topology over connective  $E_\infty$  rings)

$$QC^{perf} : \infty - \text{stacks} \rightarrow \hat{Cat}_{\infty, st}$$

obtained by Kan extension along the Yoneda embedding.

The objects of  $\text{Stab}(\mathcal{C}_A^{op})^\omega$  will be called *perfect complexes* of modules over  $A$ .

The stack  $QC^{perf}$  is key to understanding the question of compact generation of the stable  $\infty$ -category  $QC(X)$ . We need that  $QC^{perf}$  has an obstruction theory. In order for this we need to establish two things about  $QC^{perf}$

- $QC^{perf}$  is infinitesimally cohesive
- $QC^{perf}$  has a cotangent complex

It follows from a result of Toën-Vessozzi [6] that it is enough to show that

- $QC^{perf}$  is infinitesimally cohesive
- The diagonal  $\Delta : QC^{perf} \rightarrow QC^{perf} \times QC^{perf}$  is  $n$ -geometric for some  $n$ .

The first follows from the fact that  $QC$  is infinitesimally cohesive. For the second part, let  $A \in \mathcal{C}^{op}$  and let  $x, y$  be objects in  $QC^{perf}(\text{Spec} A)$ . In other words  $x$  and  $y$  are perfect modules over  $A$ . Let  $\Omega_{x,y}QC^{perf}$  be the pullback in the  $\infty$  category of  $\infty$ -stacks.

$$\begin{array}{ccc} \Omega_{x,y}QC^\omega & \longrightarrow & QC^\omega \\ \downarrow & & \downarrow \Delta \\ \text{Spec} A & \xrightarrow{x,y} & QC^\omega \times QC^\omega \end{array}$$

We'll show that  $\Omega_{x,y}QC^\omega$  is an algebraic  $n$ -stack ( $n$ -truncated) for some  $n$  depending on  $A, x$  and  $y$ . The proof is based on the Artin-Lurie criterion.

**Theorem 5.1.** (Lurie) *A functor  $F : \text{conn}E_\infty - \text{rings} \rightarrow S\text{Sets}$  is a derived algebraic  $n$ -stack (in Lurie's sense,  $n$ -truncated) iff the following are satisfied*

- (i)  $F$  is a sheaf in the etale topology
- (ii)  $F$  is  $\omega$ -accessible, it preserves  $\omega$ -filtered colimits
- (iii)  $F$  is nilcomplete, carries Postnikov towers to limits
- (iv)  $F$  is infinitesimally cohesive
- (v)  $F$  has a cotangent complex
- (vi)  $F$  is formally effective
- (vii) The restriction of  $F$  to discrete commutative rings factors through  $S\text{Sets}^{\leq n}$ .

We'll show the existence of the cotangent complex for  $\Omega_{x,y}QC^\omega$ . Checking the other hypotheses in the Artin-Lurie criterion are easy.

Let  $B$  be an object under  $A$  in  $\mathcal{C}^{op}$ . Then the restriction of the functor  $\Omega_{x,y}QC^\omega$  to  $\mathcal{C}_{A/}^{op}$  can be described as

$$\Omega_{x,y}QC^\omega = \text{Map}_{\text{Mod}_B}(x \otimes_A B, y \otimes_A B).$$

Use the notation  $\mathcal{F} = \Omega_{x,y}QC^\omega_{/\text{Spec}A} : \mathcal{C}_{A/}^{op} \rightarrow \text{SSet}$  for the restriction of the functor  $\Omega_{x,y}QC^\omega : \mathcal{C}^{op} \rightarrow \text{SSet}$  along the natural functor  $\mathcal{C}_{A/}^{op} \rightarrow \mathcal{C}^{op}$ . The structure morphism  $\text{Spec}A \rightarrow \text{Spec}S$  has a cotangent complex. Therefore in order to show that  $\Omega_{x,y}QC^\omega$  has an absolute cotangent complex it is sufficient to show that  $\Omega_{x,y}QC^\omega \rightarrow \text{Spec}A$  has a relative cotangent complex, which is simply the cotangent complex of  $F$ .

Let  $B \in \mathcal{C}_{A/}^{op}$ , an object in  $\mathcal{C}\text{-stacks}_{/\text{Spec}A}$ . Let  $z : \text{Spec}B \rightarrow \mathcal{F}$  a map in  $\mathcal{C}\text{-stacks}_{/\text{Spec}A}$ . We want to show that the functor  $\text{Def}_{\Omega_{x,y}QC^\omega/A}(x, -) : \text{Mod}_B \rightarrow \text{SSet}$  is corepresentable. Recall this is equivalent to the functor  $\text{Def}_{\mathcal{F}}(x, -)$ . Let  $B \oplus M$  be the trivial square-zero extension of  $B$  along  $M \in \text{Mod}_B$ . We have

$$\mathcal{F}(\text{Spec}B) = \text{Map}_{\text{Mod}_B}(x \otimes_A B, y \otimes_A B) \simeq \text{Hom}_{\text{Mod}_A}(x, y \otimes_A B)$$

$$\mathcal{F}(\text{Spec}(B \oplus M)) = \text{Map}_{\text{Mod}_{(B \oplus M)}}(x \otimes_A (B \oplus M), y \otimes_A (B \oplus M)) \simeq \text{Hom}_{\text{Mod}_A}(x, y \otimes_A (B \oplus M))$$

$$\simeq \text{Hom}_{\text{Mod}_A}(x, y \otimes_A B) \times \text{Hom}_{\text{Mod}_A}(x, y \otimes_A M)$$

All these equivalences commute with the natural map

$$\mathcal{F}(\text{Spec}(B \oplus M)) \rightarrow \mathcal{F}(\text{Spec}B).$$

Therefore the deformation space  $\text{Def}_{\mathcal{F}}(x, M)$  which is the homotopy fiber of this map at  $x$  is equivalent to  $\text{Hom}_{\text{Map}_A}(x, y \otimes_A M)$ . There is a chain of equivalences

$$\begin{aligned} \text{Def}_{\mathcal{F}}(x, M) &\simeq \text{Hom}_{\text{Mod}_A}(x, y \otimes_A M) \\ &\simeq \Omega^\infty(\text{Mor}_A(x, y) \otimes_A M) \\ &\simeq \Omega^\infty((\text{Mor}_A(x, y) \otimes_A B) \otimes_B M) \\ &\simeq \Omega^\infty((\text{Mor}_B((\text{Mor}_A(x, y) \otimes_A B)^v, M)) \\ &\simeq \text{Hom}_{\text{Mod}_B}(\text{Mor}_A(x, y) \otimes_A B)^v, M) \end{aligned}$$

The notation  $\text{Mor}_A(x, y)$  is used for  $\text{Hom}_{\text{Mod}_A}(x, y)$  when considered as an object of the stable  $\infty$ -category  $\text{Mod}_A$ .

The equivalences follow from the facts that  $\text{Mor}_A(x, y)$  is a compact object when  $x$  and  $y$  are compact,  $\text{Mod}_A$  is compactly generated under filtered colimits by  $A$  and compact objects are dualizable in  $\text{Mod}_B$ .

Therefore  $\text{Def}_{\mathcal{F}}(x, -)$  is corepresentable by the  $B$ -module  $L_{\mathcal{F},x} := (\text{Mor}_A(x, y) \otimes_A B)^v$ .

Suppose given a commutative diagram in  $\mathcal{C}\text{-stacks}/\text{Spec} A$

$$\begin{array}{ccc} \text{Spec} C & \xrightarrow{u} & \text{Spec} B \\ & \searrow w \quad \swarrow z & \\ & \mathcal{F} & \end{array}$$

we have the equivalences

$$\begin{aligned} L_{\mathcal{F},w} &\simeq (\text{Mor}_A(x, y) \otimes_A C)^v \simeq \text{Mor}_A(\text{Mor}_A(x, y), C) \\ u^* L_{\mathcal{F},z} &\simeq (\text{Mor}_A(x, y) \otimes_A B)^v \otimes C \simeq \text{Mor}_A(\text{Mor}_A(x, y), B) \otimes_C B \end{aligned}$$

The equivalences follow simply from adjunction are compatible with the natural map  $u^* L_{\mathcal{F},z} \rightarrow L_{\mathcal{F},w}$  making it an equivalence in  $\text{Mod}_C$ .

This completes the proof that  $\Omega_{x,y} QC^\omega$  has a cotangent complex. We need to verify the rest of the Artin-Lurie conditions to show that it is an algebraic stack. Then applying the proposition of [6] it follows that  $QC^\omega$  has a cotangent complex.

## 6 Proof of the Main Theorem

**Definition 6.1.** ([1]) *A derived  $\infty$ - $\mathcal{C}$ -stack  $X$  is perfect if*

- (i)  *$X$  has affine diagonal,*
- (ii)  *$QC(X)$  is a presentable stable  $\infty$ -category, or equivalently the triangulated category  $ho(QC(X))$  is compactly generated.*

Suppose  $A_\bullet$  is a cosimplicial object in  $\mathcal{C}^{op}$  which is level-wise truncated as objects in the  $\infty$ -category  $\mathcal{C}^{op}$ . Then the derived algebraic  $\mathcal{C}$ -stack  $X = \text{colim}_{\Delta^{op}} \text{Spec}(A_\bullet)$  can be obtained as finitely many square-zero extensions of the (non-derived) *classical* algebraic  $\infty$ - $\mathcal{C}$ -stack

$$X^{cl} = \text{colim}_{\Delta^{op}} (\text{Spec}(\pi_0 A_\bullet))$$

There is a natural map  $i : X^{cl} \rightarrow X$ . Suppose we know that  $X^{cl}$  is perfect, what can be said about the perfectness of derived counterpart  $X$ ? Since  $X^{cl} \rightarrow X$  is an infinitesimal extension of stacks, we shall consider the following question: suppose  $i : X \rightarrow \tilde{X}$  is a square-zero extension of an  $\infty$ -algebraic stack  $X$  and suppose  $QC(X)$  is compactly generated. What can be said about the presentability of the stable category  $QC(\tilde{X})$ ?

- (I) We've seen in the previous section that  $QC^\omega$  has an obstruction theory. Therefore we can use  $L_{QC^\omega}$  to lift the compact objects in  $QC(X)$  to compact objects in  $QC(\tilde{X})$ . The space of all such lifts is a deformation space

$$\begin{array}{ccc}
& & \tilde{X} \\
& \nearrow i & \downarrow \tilde{u} \\
X & \xrightarrow{u} & QC^\omega
\end{array}$$

There is an obstruction in the Andre-Quillen cohomology group

$$\alpha(u) \in \mathrm{Hom}_{\mathrm{Mod}_{\mathcal{O}_X}}(u^* L_{QC^\omega}, N)$$

(where  $N \in \mathrm{Stab}(\mathcal{C}_{/A_\bullet}^{op})$  is a  $\mathcal{O}_X$ -module, so that  $\tilde{X} = \mathrm{colimSpec}(A_\bullet^?)$  for some derivation  $\eta : L_X \rightarrow N$ ). If  $\alpha(u) = 0$  let  $\tilde{u}$  be a deformation of  $u$ .

- (II) Given  $x \in QC(\tilde{X})$ . Then  $i^*(x) \in QC(X)$ . Since  $QC(X)$  is compactly generated, there exists  $u \in QC(X)^\omega$  and a non-zero map  $u \rightarrow i^*(x)$  in  $QC(X)$ . We want to know if there is a lift of the map  $f : u \rightarrow i^*(x)$  in  $QC(X)$  to an map  $\tilde{u} \rightarrow x$  in  $QC(\tilde{X})$  under the map of stable  $\infty$ -categories

$$QC(\tilde{X}) \rightarrow QC(X)$$

induced by the natural map  $i : X \rightarrow \tilde{X}$ .

The space of all possible lifts is the space of deformations of the map  $u \rightarrow i^*(x)$  and is controlled by the cotangent complex of the  $\infty$ -stack  $\Omega_{u, i^*x} QC$ .

We'll give an description of the space of lifts of the map  $f : u \rightarrow i^*(x)$  to  $\tilde{f} : \tilde{u} \rightarrow x$  in  $QC(\tilde{X})$ .

That  $\Omega_{u, i^*(x)} QC \simeq X \times_{QC} X$  in the category of  $\infty$ -stacks means that for any affine  $\mathcal{C}$ -scheme  $\Omega_{u, i^*(x)} QC(\mathrm{Spec} A)$  is the  $\infty$  category  $\mathrm{Hom}_{\infty\text{-stacks}}(\mathrm{Spec} A, \Omega_{u, i^*x} QC)$  in which the 0-simplices are triplets  $(f, g, \phi)$  where

$$f, g : \mathrm{Spec} A \rightarrow X$$

and

$$\phi : f^*u \rightarrow g^*i^*(x)$$

is a map in  $\mathrm{Mod}_A$ . The 1-cells are morphisms between such triplets defined in the natural way.

In particular, the if we take the test space to be  $X$  itself and a square zero-extension  $\tilde{X}$  of  $X$ , then the mapping spaces are

$$\Omega_{u, i^*x} QC(X) = \mathrm{Hom}_{\infty\text{-St}}(X, X \times_{QC} X)$$

$$\Omega_{u, i^*x} QC(\tilde{X}) = \mathrm{Hom}_{\infty\text{-St}}(\tilde{X}, X \times_{QC} X)$$

The first space is the  $\infty$ -category whose objects are triplets  $(f, g : X \rightarrow X, \phi : f^*x \rightarrow g^*y \in \mathrm{Mod}_{\mathcal{O}_X})$ . The second space is the  $\infty$ -category whose objects are triplets  $(f', g' : \tilde{X} \rightarrow X, \phi' : f'^*x \rightarrow g'^*y \in \mathrm{Mod}_{\mathcal{O}_{\tilde{X}}})$ . Here  $f'^*x$ ,  $g'^*y$  and  $\phi'$  are *not* deformations of  $f^*x$ ,  $g^*y$  and  $\phi$  respectively.

However if we consider the point in  $\Omega_{u,i^*x}QC(X)$  represented by the object  $(1, 1, f)$  corresponds to the triplet  $(x, y, f : u \rightarrow i^*x)$  in  $X \times_{QC} X$ , then the fiber of  $\Omega_{u,i^*x}QC(\tilde{X}) \rightarrow \Omega_{u,i^*x}QC(X)$  over this point

$$\begin{array}{ccc} \mathfrak{D} & \longrightarrow & \Omega_{u,i^*x}QC(\tilde{X}) \\ \downarrow & & \downarrow \\ \star & \xrightarrow{(1,1,f)} & \Omega_{u,i^*x}QC(X) \end{array}$$

is the  $\infty$ -category of objects  $(u', x', \tilde{f})$  which are respectively deformations of  $x, u$  and  $f : u \rightarrow i^*(x)$  to  $QC(\tilde{X})$ . This deformation space is larger than the one we need. We want the space of deformations of the map  $f$  that keeps a fixed choice of deformations of the source  $u$  and target  $i^*x$ .

Consider the moduli functor  $\mathcal{F} : \infty\text{-stacks}_{/X \times X} \rightarrow Cat_\infty$  obtained by restricting  $\Omega_{u,i^*x}QC$  along the natural functor  $\infty\text{-stacks}_{/X \times X} \rightarrow \infty\text{-stacks}$ . Let  $z : \text{Spec}A \rightarrow \mathcal{F}$  be a map in  $\infty\text{-St}_{/X \times X}$ . Then the mapping space

$$\mathcal{F}(\text{Spec}A)$$

is the  $\infty$ -category whose objects are maps  $\phi : f^*x \rightarrow g^*y$  in  $\text{Mod}_{\mathcal{O}_X}$ . Here  $f, g : \text{Spec}A \rightarrow X \times X$  is the test space in  $\infty\text{-St}_{X \times X}$ . Denote this test space by  $\text{Spec}A_{f,g}$ .

$X$  is naturally an object in  $\infty\text{-St}_{/X \times X}$  via the identity maps. We will denote this version of  $X \in \infty\text{-St}_{/X \times X}$  by  $X_{1,1}$ .

Let  $\tilde{X}$  be considered an object in  $\infty\text{-St}_{/X \times X}$  via the derivations  $d_1, d_2 : \tilde{X} \rightarrow X \times X$  so that  $d_1^*u = \tilde{u}$  and  $d_2^*(i^*x) = x$ . Denote this object of the over category by  $\tilde{X}_{d_1,d_2}$ .

Now consider the point in  $\mathcal{F}(X_{1,1})$  corresponding to the map  $f : u \rightarrow i^*x$ . The fiber of the natural map  $\mathcal{F}(\tilde{X}_{d_1,d_2}) \rightarrow \mathcal{F}(X_{1,1})$  over this point

$$\begin{array}{ccc} \mathfrak{D}_0 & \longrightarrow & \mathcal{F}(\tilde{X}_{d_1,d_2}) \\ \downarrow & & \downarrow \\ \star & \xrightarrow{f} & \mathcal{F}(X_{1,1}) \end{array}$$

is the  $\infty$ -category whose objects are exactly the deformations of the map  $f : u \rightarrow i^*x$  in  $QC(X)$  to  $\tilde{f} : \tilde{u} \rightarrow x$  in  $QC(\tilde{X})$ .

Therefore if  $\mathcal{F}$  has an obstruction theory, this deformation problem of lifting the map  $f$

$$\begin{array}{ccc} & & \tilde{X}_{d_1,d_2} \\ & \nearrow i & \downarrow \tilde{f} \\ X_{1,1} & \xrightarrow{f} & \mathcal{F} \end{array}$$



is controlled by the cotangent complex  $L_{\mathcal{F}}$ . Recall that this is equivalent to the relative cotangent complex  $L_{\Omega_{u,i^*x}QC/X \times X}$  with respect to the natural map  $\Omega_{u,i^*x}QC = X \times_{QC} X \rightarrow X \times X$  of  $\infty$ -stacks.

More precisely, there is a cohomological obstruction

$$\beta(f) \in \mathrm{Hom}_{\mathcal{O}_{X_{1,1}}}(f^*L_{\mathcal{F}}, N)$$

Alternately this obstruction lives in

$$\mathrm{Hom}_{\mathcal{O}_X}((1, 1, f)^*L_{\Omega_{u,i^*x}QC}, N).$$

If  $\beta(f) = 0$  there exists deformations of  $f$ . The space of all possible deformations  $\tilde{f} : \tilde{u} \rightarrow x$  is

$$\Omega\mathrm{Hom}_{\mathcal{O}_X}((1, 1, f)^*L_{\Omega_{u,i^*x}QC}, N).$$

For these two steps to work we need the two moduli stacks  $QC^\omega$  and  $\Omega_{u,i^*x}QC$  *have deformation theory*. In other words that they are infinitesimally cohesive and have cotangent complexes. This has already been established for  $QC^\omega$ . Checking that the second space is infinitesimally cohesive is formal. Now we come to the existence of the cotangent complex for  $\Omega_{u,i^*x}QC$ .

Since  $i^*x$  need not be *compact*,  $\Omega_{u,i^*x}QC$  *does not* have a cotangent complex in general.

However  $i^*x \in \mathrm{Mod}_{\mathcal{O}_X}$  and  $X$  is perfect. Therefore  $i^*x$  is a filtered colimit of perfect modules over  $\mathcal{O}_X$ . Let us suppose that  $i^*x = \mathrm{colim} y_\beta$ , for  $\beta : X \rightarrow QC^\omega$ . Then the natural map

$$\mathrm{Hom}_{\mathcal{O}_X}(u, \mathrm{colim} y_\alpha) \rightarrow \mathrm{colim} \mathrm{Hom}_{\mathcal{O}_X}(u, y_\alpha)$$

is an equivalence since  $u$  is compact. Therefore any map  $f : u \rightarrow \mathrm{colim} y_\alpha = i^*x$  factors through  $\beta : u \rightarrow y_\beta$  for some  $\beta$ .

Since  $d_2^*$  is a left adjoint, it preserves colimits,

$$x = d_2^*(\mathrm{colim} y_\alpha) \simeq \mathrm{colim}(d_2^*y_\alpha)$$

It is clear that  $d_2^*y_\alpha$  need not be compact.

Replace the moduli stacks  $\Omega_{u,i^*x}QC$  in the second step with  $\Omega_{u,y_\beta}QC$ . This one does indeed have an obstruction theory. This means that the functor

$$\mathcal{G} : \mathcal{C}_{/X \times X} \rightarrow \widehat{Cat}_\infty$$

is infinitesimally cohesive and has a cotangent complex.

There exists a natural obstruction in the Andre-Quillen cohomology

$$\alpha(u, y_\beta) \in \mathrm{Hom}_{\mathcal{O}_X}(\beta^*L_{\mathcal{G}}, N)$$

for lifting the map  $\beta : u \rightarrow y_\beta$  to  $\tilde{\beta} : \tilde{u} \rightarrow d_2^*(y_\beta)$ . The space of all such deformations is equivalent to the space

$$\Omega\mathrm{Hom}_{\mathcal{O}_X}(\beta^*L_{\mathcal{G}}, N)$$

with loops based at the trivial derivation.

$i = \mathrm{colim} d_2^*(y_\beta)$  implies there is a unique map  $d_2^*(y_\beta) \rightarrow x$ . Compose this with  $\tilde{\beta}$  to obtain the desired lift of  $u \rightarrow i^*x$  to  $\tilde{X}$ .

## References

- [1] **David Ben-Zvi, John Francis, David Nadler**, *Integral Transforms and Drinfeld Centers in Derived Algebraic Geometry* (arXiv:0805.0157)
- [2] **Jacob Lurie**, *Derived Algebraic Geometry IV*
- [3] **Jacob Lurie**, *Higher Topos Theory*, Annals of Mathematics Studies. (arXiv:math/0608040)
- [4] **A. Neeman**, *The Grothendieck duality via Bousfield's technique's and Brown representability* (arXiv:alg-geom/9412022)
- [5] **A. Neeman**, *The connection between the K-theory localization theorem of Thomason, Trobaugh and Yao and the smashing subcategories of Bousfield and Ravenel*. Annales scientifiques de l.N.S. 4e srie, tome 25, no 5 (1992)
- [6] **B. Toën, G. Vezzosi**, *Homotopical Algebraic Geometry II*. (arXiv:math/0404373)